

GEODESICS AND NODAL SETS OF LAPLACE EIGENFUNCTIONS ON HYPERBOLIC MANIFOLDS

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ABSTRACT. Let X be a manifold equipped with a complete Riemannian metric of constant negative curvature and finite volume. We demonstrate the finiteness of the collection of totally geodesic immersed hypersurfaces in X that lie in the zero level set of some Laplace eigenfunction. For surfaces, we show that the number can be bounded just in terms of the area of the surface. We also provide constructions of geodesics in hyperbolic surfaces that lie in a nodal set but that do not lie in the fixed point set of a reflection symmetry.

1. INTRODUCTION

The structure of the zero level set—or *nodal set*—of eigenfunctions of the Laplacian on Riemannian manifolds is a topic of continuing interest [Zelditch13]. Among all manifolds, the surfaces of constant negative curvature remain an important testing ground [GRS13]. Using a lemma of [TthZld09], Jung [Jung14] showed that each closed curve of nonzero constant geodesic curvature intersects each nodal set in a finite set and, moreover, estimated the size of the intersection in terms of the eigenvalue.¹

On the other hand, it is well-known that closed curves of zero geodesic curvature, geodesics, may lie in the zero level set of an eigenfunction. Indeed, if a closed Riemannian manifold admits a reflection self-isometry r , then the fixed point set of r is totally geodesic, and infinitely many eigenfunctions vanish on the fixed point set.

Let $N(X)$ denote the cardinality of the collection of totally geodesic immersed hypersurfaces M in X such that there exists a Laplace eigenfunction φ so that either $\varphi|_M = 0$ or $\nu\varphi = 0$ for each vector ν normal to M . In this note we prove

Theorem 1.1. *If X is a finite volume hyperbolic manifold, then $N(X)$ is finite.*

In contrast, if X is the standard sphere or flat torus, then $N(X)$ is the first uncountable ordinal. Indeed, if the isometry group of a compact Riemannian manifold contains a one parameter subgroup K and a reflection that does not commute with K , then for each $k \in K$ the fixed point set of $k \circ r \circ k^{-1}$ is a totally geodesic hypersurface that belongs to the nodal set of an eigenfunction.

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¹Only closed horocycles and boundaries of disks were considered in [Jung14], but the methods also apply to curves consisting of points that are equidistant to a closed geodesic.

In §2, we prove Theorem 1.1. We use the Schwarz reflection principle to show that $N(X)$ is bounded above by the number $N'(X)$ of totally geodesic immersed hypersurfaces M such that there exists an orbifold covering $p : X \rightarrow Y$ such that $p(M)$ lies in the boundary of Y . We then use a theorem of H.-C. Wang [Wan67] and a compactness argument to show that $N'(X)$ is finite. In §3, we address the question as to whether one can bound $N(X)$ in terms of the volume of X . We answer affirmatively in the case where X is two-dimensional. In §4, we construct some nodal sets on hyperbolic surfaces that do not belong to the fixed point set of a reflection isometry. In §5, we raise some questions.

2. THE PROOF OF THEOREM 1.1

Recall that a hyperbolic manifold is a quotient of hyperbolic space \mathbf{H}^n by a torsion-free discrete subgroup of the Lie group, $\text{Isom}(\mathbf{H}^n)$, self-isometries of \mathbf{H}^n . More generally, we will consider quotients by discrete subgroups of $\text{Isom}(\mathbf{H}^n)$ that include finite order elements including reflections.

The group $\text{Isom}(\mathbf{H}^n)$ acts on functions $f : \mathbf{H}^n \rightarrow \mathbb{R}$ by precomposition: $(f, g) \mapsto f \circ g$. Let $\Gamma_f \subset \text{Isom}(\mathbf{H}^n)$ denote the set of $g \in \text{Isom}(\mathbf{H}^n)$ so that either $f \circ g = f$ or $f \circ g = -f$. If f is continuous, then Γ_f is closed.

Lemma 2.1. *If f is a nonconstant continuous function such that Γ_f contains a cofinite discrete subgroup $\Gamma \subset \text{Isom}(\mathbf{H}^n)$, then Γ_f is also discrete.*

Proof. Suppose to the contrary that Γ_f is not discrete. Then since it is a closed subgroup of a Lie group, it contains a one parameter subgroup G . Since Γ is cofinite, its limit set in the ideal boundary of \mathbf{H}^n is dense. It follows that the group generated by the conjugates $\gamma G \gamma^{-1} \neq G$, $\gamma \in \Gamma$, act transitively on \mathbf{H}^n . In particular, Γ_f acts transitively and so f is constant. \square

Remark 2.2. Lemma 2.1 is similar to an observation made about Laplace eigenfunctions in the last paragraph of section 2.2 of [GRS13].

If $f : \mathbf{H}^n \rightarrow \mathbb{R}$ is an eigenfunction of the Laplacian that vanishes on a totally geodesic hypersurface $M \subset \mathbf{H}^n$, then the reflection $r_M \in \text{Isom}(\mathbf{H}^n)$ about M belongs to Γ_f . This is a well-known variant of the Schwarz reflection principle. Here is a sketch of the proof: Since the Laplacian commutes with the action by isometries, the function $f \circ r_M$ is an eigenfunction of Δ with the same eigenvalue. Define $h : \mathbf{H}^n \rightarrow \mathbb{R}$ by setting h equal to f on one component of $\mathbf{H}^n - M$ and equal to $-f \circ r_M$ on the other component. Because both f and $-f \circ r_M$ vanish on M , the function h belongs to the first Sobolev space and h is a weak solution to the Laplace eigenvalue problem. A well-known argument using elliptic regularity gives that h is a strong solution. But f and h agree on an open set and hence $f = h$ and $f = -f \circ r_M$. A similar argument shows that if the normal derivative of an eigenfunction f vanishes on M , then $f \circ r_M = f$.

Given a totally geodesic hypersurface $M \subset \mathbf{H}^n/\Gamma$, let \widetilde{M} denote a lift to \mathbf{H}^n . Since each lift differs by an element of Γ , the group $\Gamma(\widetilde{M})$ generated by Γ and $r_{\widetilde{M}}$ does not depend on the lift of M .

Suppose that a nonconstant eigenfunction f (or its normal derivative) vanishes on a totally geodesic immersed hypersurface M . Since its lift, \widetilde{f} , to \mathbf{H}^n is nonconstant, Lemma 2.1 implies that $\Gamma_{\widetilde{f}}$ is discrete and hence so is $\Gamma(M)$. Therefore, $N(X)$ is at

most the number $N'(X)$ of totally geodesic immersed hypersurfaces M such that $\Gamma(M)$ is discrete.

In order to show that $N'(X)$ is finite, we first note that by a theorem of Wang [Wang67], the cofinite group Γ is contained in at most finitely many discrete subgroups of $\text{Isom}(\mathbb{H}^n)$. Thus, the number of possible discrete groups of the form $\Gamma(M)$ is finite. Thus, the following proposition completes the proof of Theorem 1.1.

Proposition 2.3. *If Γ is a cofinite discrete subgroup of $\text{Isom}(\mathbf{H}^n)$ and $\Gamma^* \supset \Gamma$ is discrete, then only finitely many totally geodesic immersed hypersurfaces $M \subset \mathbf{H}^n/\Gamma$ satisfy $\Gamma(M) = \Gamma^*$.*

Proof. First we note that there is a compact subset $K \subset \mathbf{H}^n/\Gamma$ such that each totally geodesic immersed submanifold $M \subset \mathbf{H}^n/\Gamma$ must intersect K . Indeed, let K be the complement of a union of disjoint horoball neighborhoods of the cusps. Any lift to \mathbf{H}^n of a geodesic on M joins two distinct points on the ideal boundary. If an endpoint does lie at the end of a cusp, then it enters K ; and if both endpoints lie in the closure of horoballs, they lie in distinct horoballs and hence must enter K .

Suppose that M_k is an infinite sequence of distinct totally geodesic immersed hypersurfaces with $\Gamma(M_k) = \Gamma^*$ for each k . Since each M_k intersects K , there exists an orthonormal frame F_n above some point in K such that the first $n - 1$ vectors in the frame belong to TM . Since K is compact, so is the frame bundle over K , and hence we may extract a subsequence converging to a frame F_∞ . Exponentiation of the space spanned by the first $n - 1$ vectors in F_∞ determines a totally geodesic immersed hypersurface M_∞ . Lift M_∞ and the M_n to totally geodesic hypersurfaces \widetilde{M}_∞ and \widetilde{M}_k in \mathbf{H}^n so that \widetilde{M}_k converges to \widetilde{M}_∞ uniformly on compact subsets. It follows that the sequence of distinct elements $r_{\widetilde{M}_k}$ converges to $r_{\widetilde{M}_\infty}$. This contradicts the discreteness of Γ^* . \square

3. TOWARDS BOUNDING $N(X)$ IN TERMS OF THE VOLUME OF X

It is natural to ask whether $N(X)$ can be bounded in terms of the volume of X . As seen in the previous section, the finiteness of $N'(X)$ —and hence $N(X)$ —is obtained combining the main theorem of [Wang67] and Proposition 2.3. As far as the authors are aware, the main theorem of [Wang67] has not yet been made quantitative, and, moreover, we do not know how to quantify Proposition 2.3 for general n . Nonetheless, in this section, we will show that the number of distinct $\Gamma(M)$ can be bounded in terms of $\text{Vol}(M)$ and also that Proposition 2.3 can be quantified if $n = 2$. In particular, we will obtain

Theorem 3.1. *For each $A > 0$, there exists C_A such that if X is a hyperbolic surface with $\text{area}(X) < A$, then $N(X) \leq N'(X) < C_A$.*

We begin with a bound of the number $N^*(X)$ of distinct discrete groups $\Gamma(M)$ that arise from totally geodesic immersed hypersurfaces $M \subset X$.

Theorem 3.2. *For each $V > 0$ there exists a constant C_V such that if X is an n -dimensional hyperbolic manifold with volume less than V , then $N^*(X) < C_V$.*

Proof. Let $X = \mathbb{H}/\Gamma$, let r be the reflection in a lift of a totally geodesic immersed hypersurface M , and suppose that $\Gamma(M) = \langle \Gamma, r \rangle$ is discrete. Both Γ and $r\Gamma r^{-1}$ are of finite index in $\Gamma(M) = \langle \Gamma, r \rangle$, and hence $\Gamma \cap r\Gamma r^{-1}$ is of finite index in

both Γ and $r\Gamma r^{-1}$. In other words, r lies in the commensurator $\text{Comm}(\Gamma)$ of Γ [McL-Reid]. In particular, $\Gamma(M)$ is a subgroup of $\text{Comm}(\Gamma)$ with index at most $d := [\text{Comm}(\Gamma) : \Gamma]$.

Suppose that d is finite. Then $\text{Comm}(\Gamma)$ is cofinite and d equals $\text{Vol}(X)$ divided by the covolume of $\text{Comm}(\Gamma)$. By a result of Gelandner [Gelandner11], there is a constant $K_n > 0$ such that for each discrete subgroup $\Gamma' \subset \text{Isom}(\mathbf{H}^n)$ the number of generators of Γ' is at most K_n times the covolume of Γ' . Hence, since the covolume of $\text{Comm}(\Gamma)$ is at most $\text{Vol}(X)$, the number k of generators of $\text{Comm}(\Gamma)$ is at most $K_n \cdot \text{Vol}(X)$. Also, since $k \geq 1$, the covolume of $\text{Comm}(\Gamma)$ is at least $1/K_n$. It follows that d is at most $K_n \cdot \text{Vol}(X)$. The number of subgroups of index at most d in a group generated by k elements is at most the number of subgroups of index at most d in the free group on k generators. Therefore we have a bound on $N^*(X)$ in terms of $\text{Vol}(X)$ when d is finite.

If d is not finite, then, by a well-known theorem of Margulis [McL-Reid], the lattice Γ is arithmetic. By another well-known theorem of Margulis [McL-Reid], there are only finitely many hyperbolic manifolds, X_1, X_2, \dots, X_k , with volume less than V which are quotients of arithmetic lattices. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ be lattices such that $X_i = \mathbf{H}^n/\Gamma_i$ for $i = 1, \dots, k$. By [Wang67], the number n_i of discrete groups in which Γ_i is contained is finite. Hence $N^*(X) \leq \max\{n_1, \dots, n_k\}$ for X arithmetic with volume at most V . \square

Remark 3.3. If $n \geq 4$, then the proof above can be simplified by use of a well-known result of Wang: If $n \geq 4$, then there are only finitely many hyperbolic manifolds with volume less than V (see Theorem 8.1 in [Wang72]).

Next, we bound the number of distinct M such that $\Gamma(M)$ equals a given discrete group Γ^* when $n = 2$. By combining this with Theorem 3.2, we will have Theorem 3.1.

Proposition 3.4. *Let $X = \mathbf{H}^2/\Gamma$ be a complete finite area hyperbolic surface, and let Γ^* be a cofinite subgroup of $\text{Isom}(\mathbf{H}^2)$ that contains Γ . The number of closed geodesics $M \subset X$ such that $\Gamma(M) = \Gamma^*$ is at most $\frac{173}{4\pi} \cdot \text{Area}(X)$.*

Proof. Let G be the union of the k closed geodesics $M \subset X$ such that $\Gamma(M) \subset \Gamma^*$. We have $\chi(X) = \chi(G) + \chi(X \setminus G)$, and hence by Lemma 3.5 below, we have

$$k \leq p - \chi(G) = p + \chi(X \setminus G) - \chi(X)$$

where p is the number of curves needed to divide X into pairs of pants. Since X is a finite area hyperbolic surface, we have $p \leq -(3/2) \cdot \chi(X)$ and $-\chi(X) = \text{area}(X)/2\pi$.

The Euler characteristic of $X \setminus G$ is at most the number j of connected components of $X \setminus G$ with positive Euler characteristic. A component C with $\chi(C) > 0$ is homeomorphic to a disc, and hence it lifts to a compact polygon P in \mathbf{H}^2 bounded by geodesics \widetilde{M} such that $r_{\widetilde{M}} \in \Gamma^*$. In particular, the group generated by reflections in the sides of P is a discrete subgroup of $\text{Isom}(\mathbf{H}^2)$ and hence $\text{area}(C) = \text{area}(P) \geq \pi/42$ [Siegel45]. The sum of the areas of the j components is at most $\text{area}(X)$, and so we have $j \leq \text{area}(X)/(\pi/42)$. \square

For the sake of completeness, we include the following lemma which is surely well-known. For a surface X , let $p(X)$ denote the maximal cardinality of a collection of disjoint, nonparallel, essential, nonperipheral simple closed curves on X . In other words, $p(X)$ is the number of curves needed to divide X into pairs of pants. If X is an n times punctured genus g surface, then $p(X) = 3g - 3 + n$.

Lemma 3.5. *Let X be a topological surface with $\chi(X) < 0$. If $G \subset X$ is the union of k mutually nonhomotopic, essential, nonperipheral, closed curves that meet transversally, then*

$$(1) \quad \chi(G) \leq -k + p(X).$$

Moreover, if equality holds, then a subcollection of G divides X into pairs of pants.

Proof. If a graph G and a curve γ intersect transversally, then

$$(2) \quad \chi(G \cup \gamma) \leq \chi(G) - |G \cap \gamma|.$$

Indeed, using an orientation of γ , one can map the new edges that lie in γ onto the set of intersection points $G \cap \gamma$. In particular, to each intersection point that was not a vertex in G , there is at least one new edge. If an intersection point x is not a vertex of G , then x bisects an edge of G . Hence for each new vertex, there are at least two new edges. Therefore, the Euler characteristic decreases by at least $|G \cap \gamma|$.

We prove the claim by induction on the number of connected components of G . If G is connected, then an inductive argument using (2) gives $\chi(G) \leq -k + 1$. Thus, if $p(X) \geq 1$, we have (1), and if equality holds, then $p(X) = 1$ and hence X is homeomorphic to either a once punctured torus $X_{1,0}$ or a four times punctured sphere $X_{0,4}$. A closed curve with a self-intersection point has negative Euler characteristic, and two nonhomotopic simple closed curves on either $X_{1,0}$ or $X_{0,4}$ must intersect at least once. Thus, by using induction on k , one finds that the equality implies that each closed curve in G is simple. Any essential nonperipheral simple closed curve on $X_{1,0}$ or $X_{0,4}$ divides the surface into pairs of pants.

If $p(X) = 0$, then X is homeomorphic to a pair of pants. In this case, each essential nonperipheral closed curve α has at least one self-intersection and thus $\chi(\alpha) \leq -1$. An induction argument using (2) then shows that $\chi(G) \leq -k$.

Suppose that the claim is proven when the graph has j components, and suppose that G is a graph with $j+1$ components. In other words, $G = G_+ \sqcup G_-$ where G_+ is connected and G_- has j components. If G_+ consists of a simple closed curve, then each connected component of G_- lies in a connected component X_- of $X \setminus G_+$. By the inductive hypothesis, we have $\chi(G) = \chi(G_-) \leq -(k-1) + p(X_-) = -k + p(X)$. If equality holds, then by the inductive hypothesis a subset G' of G_- divides X_- into pairs of pants, and therefore $G_+ \sqcup G'$ divides X into pairs of pants.

If G_+ is not a simple closed curve, then there exist disjoint subsurfaces X_+ and X_- so that $G_\pm \subset X_\pm$ and $\chi(X_\pm) < 0$. Thus, by the inductive hypothesis, $\chi(G) = \chi(G_+) + \chi(G_-) \leq -k + p(X_+) + p(X_-) \leq -k + p(X)$. Note that $p(X_+) + p(X_-) < p(X)$ and so equality cannot hold in this case. \square

4. SOME CONSTRUCTIONS

Not every totally geodesic component of a nodal set is the fixed point set of a reflection isometry. To see this, we construct here a closed hyperbolic surface X of genus two and an eigenfunction on X that vanishes on a closed geodesic with a point of self-intersection. Because the self-intersection is transversal, it cannot be the fixed point set of a reflection.

Let O be a regular geodesic octagon in the hyperbolic plane such that the angle at each vertex is $\pi/2$ [Buser]. See Figure 1.

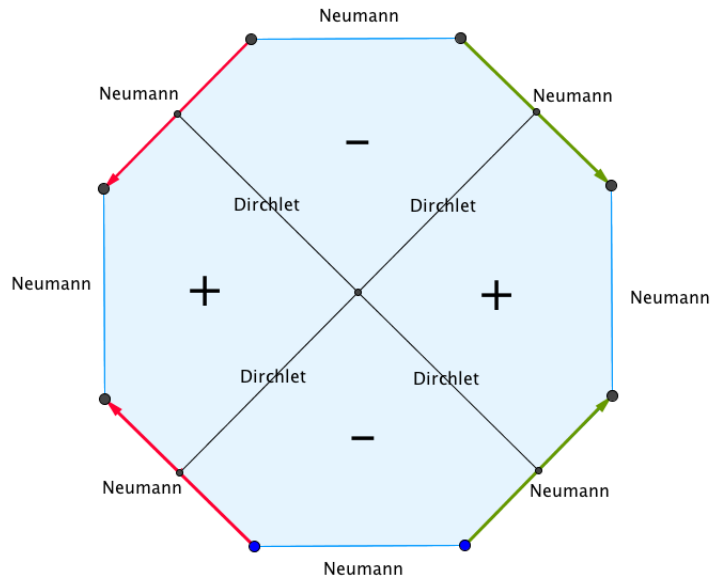


FIGURE 1

Choose two geodesics that intersect orthogonally in the center of the octagon and that bisect opposing sides. These geodesics divide the octagon into four right-angled pentagons. For one of these pentagons solve the eigenvalue problem with Dirichlet conditions on the interior geodesics and Neumann conditions on the exterior geodesics. Then extend to the other pentagons via Schwarz reflection. Because of the Neumann conditions on the exterior sides, we may identify the oriented red sides together and the oriented green sides to obtain an eigenfunction on a pair of pants such that the eigenfunction vanishes on a closed geodesic with one intersection point. By doubling the surface, we obtain an eigenfunction on a genus two surface with the desired properties.

One can also construct hyperbolic surfaces with simple closed geodesics that lie in the nodal set of an eigenfunction but that do not belong to the fixed point set of a reflective isometry. For example, let P be a hyperbolic pair of pants with equal boundary lengths [Buser]. Solve the eigenvalue problem for the Laplacian on P with Dirichlet conditions on one boundary component and Neumann boundary conditions on the other two boundary components. Glue four copies of P together as indicated in Figure 2. In particular, we use Schwarz reflection to extend the eigenfunction on P to the entire genus three surface. The eigenfunction vanishes on the simple geodesics colored red in Figure 2.

Although in the examples above the closed geodesic is not the fixed point set of a reflection isometry, each surface does have a finite covering such that the lift of the geodesic is the fixed point set of a reflection. More generally, we have the following.

Proposition 4.1. *Let X be a hyperbolic manifold with finite volume. If f is an eigenfunction that vanishes on an immersed totally geodesic hypersurface $M \subset X$,*

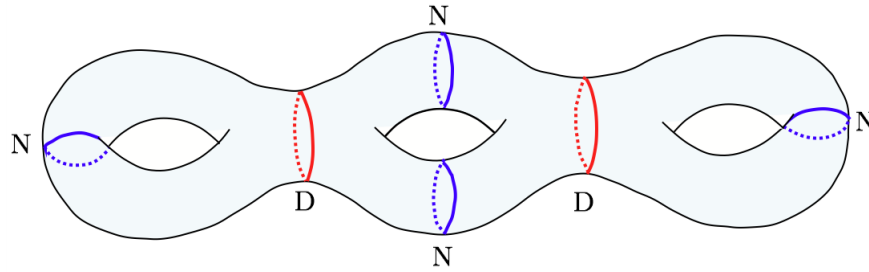


FIGURE 2

then there exists a finite cover $p : Y \rightarrow X$ and a lift N of M such that N is the fixed point set of a reflection isometry and the lift of f to Y vanishes on N .

Proof. Let Γ be a lattice such that $X = \mathbf{H}^n/\Gamma$, and let \tilde{f} be the lift of f that is invariant under Γ . We have $\Gamma \subset \langle \Gamma, r \rangle \subset \Gamma_{\tilde{f}}$ where r is the reflection through a lift \tilde{M} of M to \mathbf{H}^n . Since Γ is cofinite, the index $[\langle \Gamma, r \rangle : \Gamma]$ is finite. Hence the intersection $\Gamma' = \Gamma \cap (r\Gamma r^{-1})$ has finite index in $\langle \Gamma, r \rangle$. We have $r\Gamma' r^{-1} = \Gamma'$, and hence r descends to a reflection isometry on \mathbf{H}^n/Γ' whose fixed point set is \tilde{M}/Γ' . Since \tilde{f} vanishes on \tilde{M} , the lift of f to \mathbf{H}^n/Γ' vanishes on \tilde{M}/Γ' . \square

5. SOME QUESTIONS

In the introduction we observed that if the isometry group of a compact Riemannian manifold contains a one parameter subgroup K , then $N(X)$ may be infinite.

Question 5.1. If the isometry group of a compact Riemannian manifold is discrete, then is $N(X)$ finite?

Recently, in [EP15] it was proven that given $n \in \mathbb{N}$ and an isotopy class C of a hypersurface in a compact topological manifold M one can construct a Riemannian metric g on M and an eigenfunction φ of the Laplacian Δ_g such that φ vanishes on n disjoint hypersurfaces each belonging to the homotopy class C . The methods of [EP15] can be applied to solve the following problem: Given a compact manifold M , $\epsilon > 0$ and $n \in \mathbb{N}$, there exist a Riemannian metric g on M , n disjoint hypersurfaces H_i , and an eigenfunction φ such that each H_i is at distance at most ϵ from a totally geodesic hypersurface and $\varphi|_{H_i} = 0$. These examples do not have negative curvature.

In §3, we showed that if the dimension n of X equals 2, then one can bound $N'(X)$ and hence $N(X)$ in terms of the area of X . For $n \geq 4$, Theorem 1.1 together with Wang finiteness [Wang72] implies that there exists a bound on $N(X)$ in terms of the volume of X .²

Question 5.2. If $n = 3$, can one bound $N(X)$ in terms of the volume of X ?

In §3, we reduced Question 5.2 to a quantification of Proposition 2.3 for $n \geq 3$. A related question: To what extent does $\Gamma(M)$ determine the totally geodesic hypersurface M ?

²Thanks to the anonymous referee for reminding us to state this fact.

For each g and p , the supremum $N_{g,p}$ of $N(X)$ over the finite area hyperbolic structures on a genus g surface with p punctures is finite. One can easily construct examples that show that $N_{g,p}$ is unbounded.

Question 5.3. How does $N_{g,p}$ grow as g and/or p tend to infinity?

Finally, we wonder how general Proposition 4.1 is.

Question 5.4. If a totally geodesic hypersurface H belongs to the nodal set of an eigenfunction on a (real-analytic) Riemannian manifold (M, g) , then does there exist a Riemannian covering $(\widetilde{M}, \widetilde{g})$ and a reflection isometry of $(\widetilde{M}, \widetilde{g})$ whose fixed point set contains a lift of H ?

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